TEMPERATURE STRESSES IN THIN VISCOELASTIC PLATES WITH HEAT TRANSFER

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The solution is obtained of the quasistatic problem of thermoviscoelasticity for an unbounded plate heated by a concentrated heat source and the solution of the dynamic problem of thermoviscoelasticity for a semibounded plate heated along the boundary.

The Quasistatic Problem of Thermoviscoelasticity for an Unbounded plate with Heat Transfer. We consider a thin unbounded isotropic viscoelastic plate and we assume that heat transfer takes place through its lateral surfaces $z = \pm\delta$ with the exterior medium of zero temperature according to Newton's law. The plate is heated by an instantaneous heat source concentrated at the point $x = 0$, $y = 0$, $z = +\delta$ and having a constant capacity q, i.e., $W(x, y, z, \tau) = q\delta(x, y, z-\delta, \tau)$.

We assume that along the thickness of the plate the temperature varies linearly. In this case for the determination of the nonstationary temperature field in the plate we have the system of differential equations [1]

$$\Delta T - \varkappa^2 T = \frac{1}{a}\frac{\partial T}{\partial \tau} - Q\frac{\delta(r,\tau)}{r},$$
$$\Delta T^* - \varkappa^{*2} T^* = \frac{1}{a}\frac{\partial T^*}{\partial \tau} - Q^*\frac{\delta(r,\tau)}{r}, \qquad (1)$$

where

$$T = \frac{1}{2\delta}\int_{-\delta}^{\delta} t(r, z, \tau)\,dz; \quad T^* = \frac{3}{2\delta^2}\int_{-\delta}^{\delta} zt(r, z, \tau)\,dz;$$
$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}; \quad \varkappa^2 = \frac{\alpha}{\lambda\delta}; \quad \varkappa^{*2} = \frac{3}{\lambda\delta}\left(\alpha + \frac{2}{r^*}\right);$$
$$Q = \frac{q}{4\pi\lambda\delta}; \quad Q^* = 3Q.$$

Since the equations (1) have the same form as in the case of the heating of the plate by a linear source of heat, the fundamental solution of the heat conduction problem can be written in the known form [2]

$$T = \frac{Q}{2\tau}\exp\left(-\varkappa^2 a\tau - \frac{r^2}{4a\tau}\right),$$
$$T^* = \frac{Q^*}{2\tau}\exp\left(-\varkappa^{*2} a\tau - \frac{r^2}{4a\tau}\right). \qquad (2)$$

If the capacity of the source changes at the initial time with some quantity, which will remain constant, i.e., $W = q\delta(x, y, z-\delta)S_+(\tau)$, then the solution of the heat conduction problem will have the form:

$$T = \frac{Q}{2}K_0(\rho, \omega), \quad T^* = \frac{Q^*}{2}K_0(\rho^*, \omega^*), \qquad (3)$$

where

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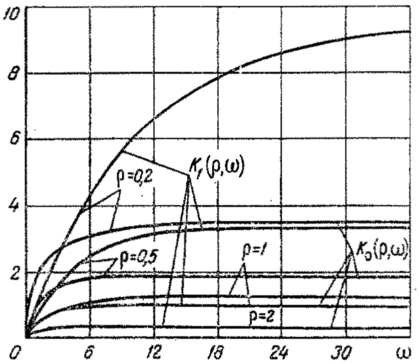

Fig. 1

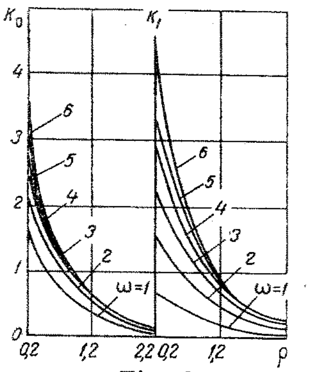

Fig. 2

Fig. 1. The graphs of $K_0(\rho, \omega)$ and $K_1(\rho, \omega)$ as functions of ω for fixed values of ρ.

Fig. 2. The graphs of $K_0(\rho, \omega)$ and $K_1(\rho, \omega)$ as functions of ρ for fixed values of ω.

$$K_m(\rho, \omega) = \int_0^{\omega} \omega^{m-1} \exp\left[-\frac{\rho}{2}\left(\omega + \frac{1}{\omega}\right)\right] d\omega, \quad m = 0, \pm 1, \pm 2, \ldots;$$

$$\rho = \varkappa r; \quad \omega = 2a\varkappa \frac{\tau}{r};$$

$$r = \sqrt{x^2 + y^2}; \quad \rho^* = \varkappa^* r; \quad \omega^* = 2a\varkappa^* \frac{\tau}{r}.$$

Recursion relations for the functions $K_m(\rho, \omega)$ are given in [2]. In order to find the numerical values of such a function of arbitrary order, it is sufficient to tabulate the functions of zero and first order. In Figs. 1 and 2 the graphs of $K_0(\rho, \omega)$ and $K_1(\rho, \omega)$ are given as functions of ω and ρ. The numerical values of the function $K_0(\rho, \omega)$ have been obtained on the basis of the known [3] values of Rykalin's function $\mu(\rho, \omega)$. For the evaluation of the function $K_1(\rho, \omega)$ we have used the Ural-1 electronic computer.

It should be mentioned that such functions are encountered in problems of heat conduction [4, 5], thermoelasticity [2, 6, 7], and thermoplasticity [8]. In the present paper, the given graphs of these functions are used for the numerical evaluation of the thermal stresses in a viscoelastic plate.

In order to determine these stresses we make use of the correspondence principle [9], i.e.,

$$\sigma_i^v = \int_0^{\tau} F(\tau - \tau_0) \frac{\partial}{\partial \tau_0} \sigma_i^E(r, \tau_0)\, d\tau_0, \tag{4}$$

where the function F for the Kelvin, Maxwell, and Biot materials, respectively, has the form:

$$F = 1 + \beta \exp(-\varkappa_1 \tau), \quad F = \exp(-\varkappa_2 \tau), \quad F = \exp(-\varkappa_0 \tau),$$

$$\beta = \frac{2(1 + \nu_{\text{к}})}{1 - 2\nu_{\text{к}}}, \quad \varkappa_1 = \frac{3}{(1 - 2\nu_{\text{к}})\vartheta^*}, \quad \varkappa_2 = \frac{E_{\text{м}}}{3G_{\text{м}}\vartheta}, \tag{5}$$

$$\vartheta = \frac{\eta}{G_{\text{м}}}, \quad \vartheta^* = \frac{\eta}{G_{\text{к}}}.$$

The thermal stresses σ_r, σ_φ in the elastic plate, induced by T (2), coincide with those caused by a linear heat source [2]

$$\sigma_r^E = \alpha_t E Q \frac{a}{r^2}\left[\exp\left(-\frac{r^2}{4a\tau}\right) - 1\right] \exp(-\varkappa^2 a\tau),$$

$$\sigma_\varphi^E = -\sigma_r^E - \frac{\alpha_t E Q}{2\tau} \exp\left(-\varkappa^2 a\tau - \frac{r^2}{4a\tau}\right). \tag{6}$$

The thermal stresses of the bending, induced in the freely supported elastic plate by T* (2), can be determined from the known [9] formulas

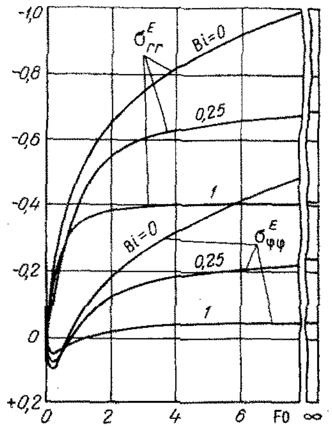

Fig. 3

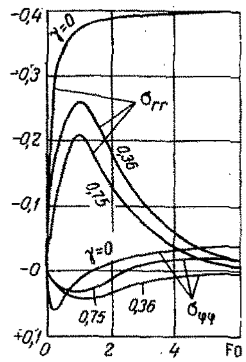

Fig. 4

Fig. 3. The variation of the radial σ^E_{rr} and annular thermal stress $\sigma^E_{\varphi\varphi}$ at the point R = 1, z = 0 of the infinite elastic plate as a function of the Biot and Fourier numbers.

Fig. 4. The variation of the radial $\overset{v}{\sigma}_{rr}$ and annular thermal stress $\overset{v}{\sigma}_{\varphi\varphi}$ at the point R = 1, z = 0 of the infinite viscoelastic plate with the Fourier number for Bi = 1 and certain values of γ.

$$\sigma_r^{*E} = 2G\,\frac{z}{r}\,\frac{\partial w}{\partial r}, \quad \sigma_\varphi^{*E} = 2Gz\,\frac{\partial^2 w}{\partial r^2}, \tag{7}$$

where the deflection w satisfies the equation

$$\Delta w = -\alpha_t(1+\nu)\,\frac{T^*}{\delta}, \tag{8}$$

whose particular solution has the form

$$w = \frac{\alpha_t}{\delta}\,(1+\nu)\,Q^*\left[\mathrm{Ei}\left(-\frac{r^2}{4a\tau}\right) - 2\ln r\right]\exp(-\varkappa^2 a\tau). \tag{9}$$

Substituting (9) into the formulas (7), we can see that the thermal stresses of the bending coincide in form with (6), where instead of Q and $\varkappa^2$ one has to put $(z/\delta)Q^*$ and $\varkappa^{*2}$.

Substituting (6) into (4), we obtain the thermal stresses in a viscoelastic plate made of Kelvin and Maxwell materials:

$$\sigma_r^v = (1+\beta)\,\sigma_r^E - \beta\varkappa_1\,\frac{Na}{r^2}\left\{\exp(-\varkappa_1\tau)\int\limits_0^\tau \exp\left[(\varkappa_1 - \varkappa^2 a)\,\tau_0 - \frac{r^2}{4a\tau_0}\right]d\tau_0 - \frac{\exp(-\varkappa^2 a\tau) - \exp(-\varkappa_1\tau)}{\varkappa_1 - \varkappa^2 a}\right\},$$

$$\sigma_\varphi^v = (1+\beta)(\sigma_r^E + \sigma_\varphi^E) - \sigma_r^v + N\,\frac{\beta}{2}\,\varkappa_1\exp(-\varkappa_1\tau)\int\limits_0^\tau \exp\left[(\varkappa_1 - \varkappa^2 a)\,\tau_0 - \frac{r^2}{4a\tau_0}\right]\frac{d\tau_0}{\tau_0}; \tag{10}$$

$$\sigma_r^v = \sigma_r^E - \varkappa_2\,\frac{Na}{r^2}\left\{\exp(-\varkappa_2\tau)\int\limits_0^\tau \exp\left[(\varkappa_2 - \varkappa^2 a)\,\tau_0 - \frac{r^2}{4a\tau_0}\right]d\tau_0 - \frac{\exp(-\varkappa^2 a\tau) - \exp(-\varkappa_2\tau)}{\varkappa_2 - \varkappa^2 a}\right\},$$

$$\sigma_\varphi^v = \sigma_r^E + \sigma_\varphi^E - \sigma_r^v + N\,\frac{\varkappa_2}{2}\exp(-\varkappa_2\tau)\int\limits_0^\tau \exp\left[(\varkappa_2 - \varkappa^2 a)\,\tau_0 - \frac{r^2}{4a\tau_0}\right]\frac{d\tau_0}{\tau_0}. \tag{11}$$

For Biot viscoelastic materials one has to replace $\varkappa_2$ by $\varkappa_0$ in the formulas (11).

If $\varkappa^2 - (\varkappa_i/a) > 0$, the thermal stresses (10) and (11) can be expressed in terms of the functions $K_0(\rho_i, \omega_i)$, $K_1(\rho_i, \omega_i)$:

$$\sigma_r^v = (1+\beta)\sigma_r^E - \beta\varkappa_1 \frac{N}{\rho_1}\exp(-\varkappa_1\tau)\left\{\frac{1}{2}K_1(\rho_1,\ \omega_1) + \left[\exp\left(-\frac{\rho_1\omega_1}{2}\right) - 1\right]\rho_1^{-1}\right\},$$

$$\sigma_\varphi^v = (1+\beta)(\sigma_r^E + \sigma_\varphi^E) - \sigma_r^v + \frac{\beta\varkappa_1}{2} N\exp(-\varkappa_1\tau) K_0(\rho_1,\ \omega_1);$$

$$\sigma_r^v = \sigma_r^E - N\frac{\varkappa_2}{\rho_2}\exp(-\varkappa_2\tau)\left\{\frac{1}{2}K_1(\rho_2,\ \omega_2) + \left[\exp\left(-\frac{\rho_2\omega_2}{2}\right) - 1\right]\rho_2^{-1}\right\},$$

$$\sigma_\varphi^v = \sigma_\varphi^E + \sigma_r^E - \sigma_r^v + \varkappa_2\frac{N}{2}\exp(-\varkappa_2\tau) K_0(\rho_2,\ \omega_2), \tag{12}$$

where

$$\rho_i = r\sqrt{\varkappa^2 - \frac{\varkappa_i}{a}};\quad \omega_i = 2\sqrt{\varkappa^2 - \frac{\varkappa_i}{a}}\ \frac{a\tau}{r},\ i = 1,\ 2.$$

The thermal stresses induced in the elastic plate by T (3), are obtained by integrating (6) with respect to time from 0 to τ:

$$\sigma_r^E = \frac{N}{\sqrt{\mathrm{Bi}}\,R^2}\left[\frac{-1+\exp(-\mathrm{Mi})}{\sqrt{\mathrm{Bi}}} + \frac{R}{2}K_1(\rho,\ \omega)\right],$$

$$\sigma_\varphi^E = -\sigma_r^E - \frac{N}{2}K_0(\rho,\ \omega), \tag{13}$$

where

$$N = \alpha_t EQ;\ \ R = \frac{r}{\delta},\ \ \mathrm{Bi} = \varkappa^2\delta^2,\ \ \mathrm{Fo} = \frac{a\tau}{\delta^2},\ \ \mathrm{Mi} = \mathrm{BiFo}.$$

For Bi = 0, these formulas coincide with those given in [8].

With the formulas (13), making use of the numerical values of the functions $K_0(\rho,\ \omega)$, $K_1(\rho,\ \omega)$ (Fig. 1, 2), calculations have been carried out for the thermal stresses at the point R = 1 for z = 0 in the elastic plate as functions of the Fourier and Biot numbers, which are represented in the form of graphs in Fig. 3. In this figure $\sigma_{rr}^E = \sigma_r^E/N$, $\sigma_{\varphi\varphi}^E = \sigma_\varphi^E/N$. For the infinitely large value of the Fourier number, these results coincide with those obtained previously in [7]. As it is clear from the graphs, the heat transfer between lateral surfaces of the plate and the medium has an essential impact on the distribution of the thermal stresses.

The thermal stresses in a viscoelastic plate, induced by the temperature field (3), can be written for a Maxwell material in the form:

$$\sigma_r^v = \frac{N}{\rho_2}\exp(-\gamma\mathrm{Fo})\left\{\left[\exp\left(-\frac{\rho_2\omega_2}{2}\right) - 1\right]\rho_2^{-1} + \frac{1}{2}K_1(\rho_2,\ \omega_2)\right\},$$

$$\sigma_\varphi^v = -\sigma_r^v - \frac{N}{2}\exp(-\gamma\mathrm{Fo})K_0(\rho_2,\ \omega_2),\quad \varkappa^2 - \frac{\varkappa_2}{a} > 0. \tag{14}$$

In the case when the surface of the plate is insulated ($\alpha = 0$), formulas (14) coincide with the formulas [9] for the thermal stresses in the space, induced by a linear heat source.

In Fig. 4 we have represented the variation of the radial σ_{rr}^V and annular thermal stress $\sigma_{\varphi\varphi}^V$ at the point R = 1, z = 0 of the viscoelastic plate for Bi = 1 as a function of the viscosity coefficient of the material $\eta = E_M\delta^2/3\gamma a$.

The Dynamic Problem of Thermoviscoelasticity for a Semibounded Plate with Heat Transfer. We consider now an isotropic semibounded plate and we assume that heat transfer takes place through the lateral surfaces $z = \pm\delta$ with the exterior medium. The temperature of the boundary x = 0 of the plate varies at the initial time with some quantity t_0 which remains constant in the sequel. The nonstationary temperature field and the induced quasistatic thermal stresses for the semibounded elastic plate, whose boundary has a linear variation of temperature as function of time, are given in [10]. Consequently, the temperature field in our case has the form

$$T = \frac{t_0}{2}\left[\exp(-x\varkappa)\mathrm{erfc}\left(\frac{x}{2\sqrt{a\tau}} - \sqrt{\mathrm{Mi}}\right) + \exp(x\varkappa)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{a\tau}} + \sqrt{\mathrm{Mi}}\right)\right]. \tag{15}$$

Applying the Laplace transform, we obtain

$$\bar{T} = \frac{t_0}{s} \exp\left(-x\sqrt{\varkappa^2 + \frac{s}{a}}\right).$$

As in the case of the viscoelastic semispace [9], we find the image of the retarding thermoelastic potential of the displacements for the viscoelastic semibounded plate in the form

$$\bar{\varphi} = -\frac{t_0\bar{m}\left[\exp\left(-x\sqrt{\varkappa^2 + \frac{s}{a}}\right) - \exp(-x\bar{\sigma}s)\right]}{s\left(s^2\bar{\sigma}^2 - \frac{s}{a} - \varkappa^2\right)},$$

where

$$\bar{m} = \alpha_t(1+\bar{\nu}); \quad \bar{\sigma}^2 = \frac{\rho}{2(\bar{\lambda}+\bar{\mu})}\left(1 + \frac{\bar{\lambda}}{2\bar{\mu}}\right).$$

For a Biot viscoelastic material we have the following image of the thermal stress σ_x^v:

$$\bar{\sigma}_x^v = -t_0 m_0 \rho s \frac{\exp\left(-x\sqrt{\varkappa^2 + \frac{s}{a}}\right) - \exp\left[-x\sigma_0\sqrt{s(s+\varkappa_0)}\right]}{\sigma_0^2\left[s^2 - \left(\frac{1}{a\sigma_0^2} - \varkappa_0\right)s - \frac{\varkappa^2}{\sigma_0^2}\right]}, \tag{16}$$

where

$$\sigma_*^2 = \frac{\rho}{2(\lambda_0+\mu_0)}; \quad m_0 = \alpha_t(3\lambda_0 + 2\mu_0)\frac{\sigma_*^2}{\rho}; \quad \sigma_0^2 = \sigma_*^2\left(1 + \frac{\lambda_0}{2\mu_0}\right).$$

In (16) we go back from the image to the preimage [4]. As a result we obtain the following expression of the thermal stress:

$$\sigma_x^v = \frac{t_0 m_0 \rho}{\sigma_0^2(p_1 - p_2)}\{p_2[f(\xi, \theta; p_2) - g(\xi, \theta; p_2)] - p_1[f(\xi, \theta; p_1) - g(\xi, \theta; p_1)]\}, \tag{17}$$

where

$$f(\xi, \theta; p_i) = \frac{\exp(p_i\theta)}{2}\left\{\exp(-\xi\sqrt{B+p_i})\,\mathrm{erfc}\left[\frac{\xi}{2\sqrt{\theta}} - \sqrt{(B+p_i)\theta}\right] + \exp(\xi\sqrt{B+p_i})\,\mathrm{erfc}\left[\frac{\xi}{2\sqrt{\theta}} + \sqrt{(B+p_i)\theta}\right]\right\};$$

$$p_{1,2} = \frac{1-\varepsilon}{2} \pm \sqrt{\frac{(1-\varepsilon)^2}{4} + B}; \quad \xi = \frac{x}{a\sigma_0};$$

$$\theta = \frac{\tau}{a\sigma_0^2}; \quad \varepsilon = \varkappa_0 a\sigma_0^2;$$

$$g(\xi, \theta; p_i) = \exp(\theta p_i)\left\{\exp\left[-\xi\left(p_i + \frac{\varepsilon}{2}\right)\right]S_+(\theta - \xi)\right.$$

$$\left. + \frac{\xi\varepsilon}{2}\int_0^\theta \exp\left[-\mu\left(p_i + \frac{\varepsilon}{2}\right)\right]\frac{I_1\left(\frac{\varepsilon}{2}\sqrt{\mu^2 - \xi^2}\right)}{\sqrt{\mu^2 - \xi^2}}S_+(\mu - \xi)\,d\mu\right\}.$$

$$B = \varkappa^2 a^2 \sigma_0^2.$$

We find the thermal stress σ_y^v in the form:

$$\sigma_y^v = -2m_0\mu_0 t_0 f(\xi, \theta; -\varepsilon) + \frac{\lambda_0\sigma_*^2}{\rho}\sigma_x^v. \tag{18}$$

On the boundary $x = 0$ of the plate the thermal stresses are:

$$\sigma_x^v = 0, \quad \sigma_y^v = -2\mu_0 m_0 t_0 \exp(-\theta\varepsilon). \tag{19}$$

We see that the thermal stresses at the boundary of the viscoelastic semibounded plate with heat transfer do not differ from those corresponding to an insulated plate [9]. For $\theta \to \infty$ the stress σ_y^v tends to zero.

For $\varkappa_0 \to 0$ from (17), (18) we obtain the solution of the dynamic problem of thermoelasticity for a semibounded plate with heat transfer [11].

NOTATION

$t(x, y, z, \tau)$	is the temperature of the plate;
τ	is the time;
$r^* = 2\delta/\lambda$	is the internal thermal resistance of the plate;
α, λ, a	are the heat transfer coefficient, thermal conductivity and thermal diffusivity;
r	is the polar radius;
2δ	is the thickness of the plate;
W	is the density of the heat source;
$\delta(x)$	is the Dirac delta function;
$S_+(x)$	is the Heaviside function;
σ_i^E, σ_i^{*E} and σ_i^v, σ_i^{*v}	are the components of the thermal stresses in the elastic and viscoelastic plates induced by T and T*, respectively;
ν	is the Poisson ratio;
G	is the shear modulus;
E	is the Young modulus;
$E_i(-r^2/4a\tau)$	is the exponential integral function;
$\vartheta = \eta/G_M$	is the relaxation time;
$\vartheta^* = \eta/G_K$	is the delay time;
ν_K, G_K	are the Poisson ratio and the shear modulus for Kelvin material;
E_M, G_M	are the Young modulus and the shear modulus for Maxwell material;
α_t	is the temperature coefficient of linear expansion;
$\varkappa_0^{-1}$	is the relaxation time of the relaxation functions $\lambda(\tau) = \lambda_0 \exp(-\varkappa_0 \tau)$ and $\mu(\tau) = \mu_0 \exp(-\varkappa_0 \tau)$;
$Bi = \alpha\delta/\lambda$	is the Biot number;
$Fo = a\tau/\delta^2$	is the Fourier number;
$Mi = BiFo$	is the Mikheev number;
$I_1(x)$	is the first order modified Bessel function of the first kind;
$erfc(x) = 1 - erf(x)$	
$erf(x)$	is the probability integral;
$\overline{\varphi} = \int_0^\infty \varphi \exp(-s\tau) d\tau$	is the Laplace transform of the function φ;
$\gamma = (\varkappa_2 \delta^2/a)$;	
ρ	is the density.

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